International Journal of Theoretical Physics, Vol. 42, No. 1, January 2003 (© 2003)

Quantum Computational Logic

S. Gudder[1]

Received November 21, 2002

A quantum computational logic is constructed by employing density operators on spaces of qubits and quantum gates represented by unitary operators. It is shown that this quantum computational logic is isomorphic to the basic sequential effect algebra [0, 1].

KEY WORDS: quantum computation; quantum logic; quantum gates.

1. INTRODUCTION

In an interesting paper (Cattaneo *et al.*, in press), the authors develop a new form of quantum logic based on the theory of quantum computation (Nielsen and Chuang, 2000; Pittenger, 2001). In this presentation, the semantics are given in terms of pure states of an n-qubit quantum system and logical connectives are interpreted as quantum gates represented by unitary operators. The present paper generalizes the work of Cattaneo *et al.* (in press) to mixed states of an n-qubit quantum system. It is also shown that the resulting quantum computational logic is isomorphic to the basic sequential effect algebra [0, 1]. Sequential effect algebras were recently introduced to study the sequential action of quantum effects that are unsharp versions of quantum events (Gudder and Greechie, in press-a,b). For further details and a discussion of quantum computational semantics, we refer the reader to Cattaneo *et al.* (in press).

2. DENSITY OPERATOR COMPUTATIONAL LOGIC

In the theory of quantum computation, a *qubit* is a two-dimensional quantum system. A pure qubit state is represented by a unit vector $|\psi\rangle$ in the two-dimensional Hilbert space $\mathbb{C}^2$. Denoting the standard orthonormal basis for $\mathbb{C}^2$ by $|0\rangle = (1, 0)$, $|1\rangle = (0, 1)$ we call $\{|0\rangle, |1\rangle\}$ the *computational basis* for the qubit. We can then

[1] Department of Mathematics, University of Denver, Denver, Colorado 80208; e-mail: sgudder@math.du.edu.

write

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

where $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$. For a positive integer n, an *n-qubit* is a 2^n-dimensional quantum system. In this case, the pure states are represented by unit vectors in $\otimes^n \mathbb{C}^2 = \mathbb{C}^{2^n}$. The 2^n unit vectors of the form $|i_1\rangle \otimes \cdots \otimes |i_n\rangle$, $i_j \in \{0, 1\}$, $j = 1, \ldots n$, give the *computational basis* for an n-qubit. It is standard practice to use the notation

$$|i_1 i_2 \cdots i_n\rangle = |i_1\rangle|i_2\rangle \cdots |i_n\rangle = |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_n\rangle$$

An arbitrary pure n-qubit state $|\psi\rangle \in \mathbb{C}^{2^n}$, $||\psi|| = 1$ has the form

$$|\psi\rangle = \sum a_{i_1} \cdots_{i_n} |i_1 \cdots i_n\rangle \tag{1}$$

where $a_{i_1} \cdots_{i_n} \in \mathbb{C}$ with $\sum |a_{i_1} \cdots_{i_n}|^2 = 1$, $i_j \in \{0, 1\}$, $j = 1, \ldots, n$.

Employing (2.1) we can write

$$\begin{aligned} |\psi\rangle &= \sum a_{i_1} \cdots_{i_{n-1}} 0 |i_1 \cdots_{i_{n-1}} 0\rangle + \sum a_{i_1} \cdots_{i_{n-1}} 1 |i_1 \cdots i_{n-1} 1\rangle \\ &= |\psi_0\rangle + |\psi_1\rangle = |\tilde{\psi}_0\rangle|0\rangle + |\tilde{\psi}_1\rangle|1\rangle \end{aligned}$$

where $\psi_0 \perp \psi_1$, $||\psi_0||^2 + ||\psi_1||^2 = 1$, $||\tilde{\psi}_0|| = ||\psi_0||$, $||\tilde{\psi}_1|| = ||\psi_1||$. We call $|\psi_0\rangle$ a 0-*vector* and $|\psi_1\rangle$ a 1-*vector*. Thus, any pure n-qubit state has the unique representation as the sum of a 0-vector and a 1-vector in the computational basis. We think of a 0-vectors as having truth-value "false" and 1-vectors as having truth-value "true." Let P_i be the orthogonal projections onto the span of the i-vectors, $i = 0, 1$. Then $P_0 + P_1 = I$. Let $\mathcal{D}(\mathbb{C}^{2^n})$ be the set of density operators on $\mathbb{C}^{2^n}$ and let $\mathcal{D} = \cup_{n=1}^{\infty} \mathcal{D}(\mathbb{C}^{2^n})$. Of course, elements of $\mathcal{D}(\mathbb{C}^{2^n})$ correspond to mixed n-qubit states. For $\rho \in \mathcal{D}(\mathbb{C}^{2^n})$ we define the *probability* of ρ by

$$p(\rho) = \mathrm{tr}(P_1 \rho) = \mathrm{tr}(\rho P_1)$$

For example, for a one-dimensional projection $|\psi\rangle\langle\psi|$ we have

$$p(|\psi\rangle\langle\psi|) = \mathrm{tr}(P_1|\psi\rangle\langle\psi|P_1) = \mathrm{tr}(|\psi_1\rangle\langle\psi_1|) = ||\psi_1||^2$$

For the uniformly distributed density operator $I/2^n$ we have

$$p\left(\frac{I}{2^n}\right) = \frac{1}{2^n}\mathrm{tr}(P_1) = \frac{2^{n-1}}{2^n} = \frac{1}{2}$$

For $\rho, \sigma \in \mathcal{D}$ we write $\rho \models \sigma$ if $p(\rho) \leq p(\sigma)$. Note that $\models$ is a reflexive, transitive relation. We write $\rho \sim \sigma$ if $p(\rho) = p(\sigma)$. Thus, $\rho \sim \sigma$ if and only if $\rho \models \sigma$ and $\sigma \models \rho$. Then $\sim$ is an equivalence relation on $\mathcal{D}$ and we denote the equivalence class containing ρ by $[\rho]$. We define $[\rho] \leq [\sigma]$ if $p(\rho) \leq p(\sigma)$. Then $\leq$ is well-defined on $L = \{[\rho] : \rho \in \mathcal{D}\}$ and is a partial order relation. If $p(\rho_1) = 1$, we call $[\rho_1]$ a *tautology* and if $p(\rho_0) = 0$ we call $[\rho_0]$ an *absurdity*. Then $[\rho_0]$ and $[\rho_1]$ are the least and greatest elements of the partially ordered set L, respectively. For $[\rho] \in L$ we define $p([\rho]) = p(\rho)$.

The NOT-*gate* on $\mathbb{C}^{2^n}$ is the unitary operator N given by the $2^n \times 2^n$ matrix $N = I_{2^{n-1}} \otimes X$ where X is the Paul matrix

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

For $\rho \in \mathcal{D}(\mathbb{C}^{2^n})$ we define NOTρ by NOT$\rho = N\rho N$. Then NOT$\rho \in \mathcal{D}(\mathbb{C}^{2^n})$. Since $X^2 = I$ we have $N^2 = I$ so that NOT(NOTρ) $= \rho$.

Lemma 2.1. *(a)* $N|\psi\rangle = |\tilde{\psi}_0\rangle|1\rangle + |\tilde{\psi}_1\rangle|0\rangle$. *(b)* $NP_0N = P_1$ *and* $NP_1N = P_0$.

Proof: (a) For an arbitrary pure state $|\psi\rangle \in \mathbb{C}^{2^n}$ we have

$$\begin{aligned} N|\psi\rangle &= (1_{2^{n-1}} \otimes X)(|\tilde{\psi}_0\rangle|0\rangle + |\tilde{\psi}_1\rangle|1\rangle) = |\tilde{\psi}_0\rangle X|0\rangle + |\tilde{\psi}_1\rangle X|1\rangle \\ &= |\tilde{\psi}_0\rangle|1\rangle + |\tilde{\psi}_1\rangle|0\rangle \end{aligned}$$

(b) Applying part (a) we have

$$\begin{aligned} NP_0N|\psi\rangle &= NP_0(|\tilde{\psi}_0\rangle|1\rangle + |\tilde{\psi}_1\rangle|0\rangle) = N|\tilde{\psi}_1\rangle|0\rangle = |\tilde{\psi}_1\rangle|1\rangle \\ &= |\psi_1\rangle = P_1|\psi\rangle| \end{aligned}$$

Hence, $NP_0N = P_1$. Since $P_0 + P_1 = I$, we have

$$I = NIN = NP_0N + NP_1N = P_1 + NP_1N$$

Hence, $NP_1N = I - P_1 = P_0$. □

It follows from Lemma 2.1 that

$$\begin{aligned} p(\text{NOT}\rho) &= \text{tr}(P_1 N\rho N) = \text{tr}(NP_1N\rho) = \text{tr}(P_0\rho) \\ &= \text{tr}(\rho) - \text{tr}(P_1\rho) = 1 - p(\rho) \end{aligned}$$

We conclude that $\rho \models \sigma$ implies that NOT$\sigma \models$ NOTρ. We define $[\rho]' = [\text{NOT}\rho]$. Then $[\rho]'' = [\rho]$, $[\rho] \leq [\sigma]$ implies that $[\sigma]' \leq [\rho]'$ and $[\rho_0]' = [\rho_1]$, $[\rho_1]' = [\rho_0]$. It follows that $(L, \leq, ')$ is a bounded orthoposet that we call the *quantum computational logic* (QCL).

Following Cattaneo *et al.* (in press), we now introduce the nonclassical gate $\sqrt{\text{NOT}}$. Let M be the 2×2 unitary matrix

$$M = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

We then have $M = \sqrt{X}$ and $\sqrt{N} = I_{2^{n-1}} \otimes M$. (These operators have four square roots and we are only considering one of them.) Let Y be the Pauli matrix

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

and define the unitary $2^n \times 2^n$ matrix R by $R = I_{2^{n-1}} \otimes Y$. We now define $\sqrt{\text{NOT}}\rho = \sqrt{N}^* \rho \sqrt{N}$. Of course, $\sqrt{\text{NOT}}(\sqrt{\text{NOT}}\rho) = \text{NOT}\rho$.

Theorem 2.2. $p(\sqrt{\text{NOT}}\rho) = \frac{1}{2} + \frac{1}{2}\text{tr}(R\rho)$.

Proof: From the definition of $\sqrt{N}$ we have

$$\begin{aligned}
\sqrt{N}^*|\psi\rangle &= I_{2^{n-1}} \otimes M^*(|\tilde{\psi}_0\rangle|0\rangle + |\tilde{\psi}_1\rangle|1\rangle) = |\tilde{\psi}_0\rangle M|0\rangle + |\tilde{\psi}_1\rangle M|1\rangle \\
&= \frac{|\tilde{\psi}_0\rangle}{2}[(1-i)|0\rangle] + (1+i)|1\rangle] + \frac{|\tilde{\psi}_1\rangle}{2}[(1+i)]|0\rangle + (1-i)|1\rangle] \\
&= \frac{1}{2}[(1-i)|\tilde{\psi}_0\rangle + (1+i)|\tilde{\psi}_1\rangle]|0\rangle + \frac{1}{2}[(1+i)|\tilde{\psi}_0\rangle + (1-i)|\tilde{\psi}_1\rangle]|1\rangle
\end{aligned}$$

Hence,

$$\begin{aligned}
\sqrt{N}P_1\sqrt{N}^*|\psi\rangle &= \frac{\sqrt{N}}{2}[(1+i)|\tilde{\psi}_0\rangle + (1-i)|\tilde{\psi}_1\rangle]|1\rangle \\
&= \frac{1}{2}[(1+i)|\tilde{\psi}_0\rangle + (1-i)|\tilde{\psi}_1\rangle]M|1\rangle \\
&= \frac{1}{4}[(1+i)|\tilde{\psi}_0\rangle + (1-i)|\tilde{\psi}_1\rangle][(1-i)|0\rangle + (1+i)|1\rangle] \\
&= \frac{1}{2}|\tilde{\psi}_0\rangle|0\rangle + \frac{1}{2}|\tilde{\psi}_1\rangle|1\rangle + \frac{i}{2}|\tilde{\psi}_0\rangle|1\rangle - \frac{i}{2}|\tilde{\psi}_1\rangle|0\rangle \\
&= \frac{1}{2}|\psi\rangle + \frac{i}{2}(|\tilde{\psi}_0\rangle|1\rangle - |\tilde{\psi}_1\rangle|0\rangle) \\
&= \left(\frac{1}{2}I + \frac{1}{2}R\right)|\psi\rangle
\end{aligned}$$

We conclude that $\sqrt{N}P_1\sqrt{N}^* = \frac{1}{2}I + \frac{1}{2}R$. Thus,

$$\begin{aligned}
p(\sqrt{\text{NOT}}\rho) &= \text{tr}(P_1\sqrt{N}^*\rho\sqrt{N}) = \text{tr}(\sqrt{N}P_1\sqrt{N}^*\rho) \\
&= \frac{1}{2}\text{tr}(\rho) + \frac{1}{2}\text{tr}(R\rho) = \frac{1}{2} + \frac{1}{2}\text{tr}(R\rho)
\end{aligned}$$

□

It follows from Theorem 2.2 that

$$p(\text{NOT}\sqrt{\text{NOT}}\rho) = p(\sqrt{\text{NOT}}\text{NOT}\rho) = \frac{1}{2} - \frac{1}{2}\text{tr}(R\rho)$$

For the uniform density operator we have

$$p\left(\sqrt{\text{NOT}}\frac{I}{2^n}\right) = \frac{1}{2} + \frac{1}{2}\text{tr}\left(\frac{R}{2^n}\right) = \frac{1}{2}$$

Corollary 2.3. *For a one-dimensional projection $|\psi\rangle\langle\psi|$ we have*

$$p(\sqrt{\text{NOT}}|\psi\rangle\langle\psi|) = \frac{1}{2} - \text{Im}\,(\langle\tilde{\psi}_1 \mid \tilde{\psi}_0\rangle)$$

Proof: Applying Theorem 2.2 gives

$$\begin{aligned} p(\sqrt{\text{NOT}}|\psi\rangle\langle\psi|) &= \frac{1}{2} + \frac{1}{2}\text{tr}(R|\psi\rangle\langle\psi|) = \frac{1}{2} + \frac{1}{2}\text{tr}[R(|\tilde{\psi}_0\rangle|0\rangle + |\tilde{\psi}_1\rangle|1\rangle)\langle\psi|] \\ &= \frac{1}{2} + \frac{i}{2}\text{tr}[(|\tilde{\psi}_0\rangle|1\rangle - |\tilde{\psi}_1\rangle|0\rangle)\langle\psi|] \\ &= \frac{1}{2} + \frac{i}{2}\text{tr}[(|\tilde{\psi}_0\rangle|1\rangle - |\tilde{\psi}_1\rangle|0\rangle)(\langle\tilde{\psi}_0|\langle 0| + \langle\tilde{\psi}_1|\langle 1|)] \\ &= \frac{1}{2} + \frac{i}{2}[\langle\tilde{\psi}_1 \mid \tilde{\psi}_0\rangle - \langle\tilde{\psi}_0 \mid \tilde{\psi}_1\rangle] \\ &= \frac{1}{2} - \text{Im}(\langle\tilde{\psi}_1 \mid \tilde{\psi}_0\rangle) \end{aligned}$$

□

The *quantum Toffoli gate* $T^{m,n,1} : \mathbb{C}^{2^{(m+n+1)}} \to \mathbb{C}^{2^{(m+n+1)}}$ is the unitary operator given by

$$T^{(m,n,1)}|i_1 \cdots i_m j_1 \cdots j_n k\rangle = |i_1 \cdots i_m j_1 \cdots j_n\rangle \mid (i_m \cdot j_n + k)\,(\text{mod}\,2)\rangle$$

For $\rho \in \mathcal{D}(\mathbb{C}^{2^m})$, $\sigma \in \mathcal{D}(\mathbb{C}^{2^n})$, following Cattaneo *et al.* (in press) we define AND $(\rho, \sigma) \in \mathcal{D}(\mathbb{C}^{2^{(m+n+1)}})$ by

$$\text{AND}(\rho, \sigma) = T^{(m,n,1)}\rho \otimes \sigma \otimes |0\rangle\langle 0|T^{(m,n,1)}$$

On the QCL L we define AND $([\rho], [\sigma])$ as $[\text{AND}\,(\rho, \sigma)]$. It follows from our next result that this is well-defined. We denote the projection P_1 on $\mathbb{C}^{2^n}$ by $P_1^{(n)}$.

Theorem 2.4. *(a) For $\rho \in \mathcal{D}(\mathbb{C}^{2^m})$, $\sigma \in \mathcal{D}(\mathbb{C}^{2^n})$ we have*

$$T^{(m,n,1)} P_1 T^{(m,n,1)} \rho \otimes \sigma \otimes |0\rangle\langle 0| = P_1^{(m)} \rho \otimes P_1^{(n)} \sigma \otimes |0\rangle\langle 0|$$

(b) For $\rho, \sigma \in \mathcal{D}$ we have p (AND $(\rho, \sigma)) = p(\rho)p(\sigma)$.

Proof: (a) Since

$$\begin{aligned} T^{(m,n,1)}\ & P_1 T^{(m,n,1)} \rho \otimes \sigma \otimes |0\rangle\langle 0||i_1 \cdots i_m j_1 \cdots j_n 0\rangle \\ &= T^{(m,n,1)} P_1 T^{(m,n,1)} \rho |i_1 \cdots i_m\rangle \sigma | j_1 \cdots j_n\rangle |0\rangle \\ &= P_1^{(m)} \rho |i_1 \cdots i_m\rangle P_1^{(n)} \sigma | j_1 \cdots j_n\rangle |0\rangle \\ &= P_1^{(m)} \rho \otimes P_1^{(n)} \sigma |0\rangle\langle 0||i_1 \cdots i_m j_1 \cdots j_n 0\rangle \end{aligned}$$

the result now follows. (b) For $\rho, \sigma \in \mathcal{D}$ we have by part (a) that

$$\begin{aligned} p(\text{AND}(\rho, \sigma)) &= \operatorname{tr}(T^{(m,n,1)} P_1 T^{(m,n,1)} \rho \otimes \sigma \otimes |0\rangle\langle 0|) \\ &= \operatorname{tr}\big(P_1^{(m)} \rho \otimes P_1^{(n)} \sigma \otimes |0\rangle\langle 0|\big) \\ &= \operatorname{tr}\big(P_1^{(m)} \rho\big) \operatorname{tr}\big(P_1^{(n)} \sigma\big) = p(\rho)p(\sigma) \end{aligned}$$

□

For $\rho, \sigma \in \mathcal{D}$ we define

$$\text{OR}(\rho, \sigma) = \text{NOT}[\text{AND}(\text{NOT}\rho, \text{NOT}\sigma)]$$

Applying Theorem 2.4(b) we have

$$p(\text{OR}(\rho, \sigma)) = p(\rho) + p(\sigma) - p(\rho)p(\sigma)$$

On the QCL L we define OR $([\rho], [\sigma]) = [\text{OR}(\rho, \sigma)]$ and the previous equations shows that this is well-defined. In summary, we have defined the logical connectives NOT, AND, and OR on the QCL L.

3. SEQUENTIAL EFFECT ALGEBRAS

Effect algebras (Dvurečenskij and Pulmannová, 2000; Foulis and Bennett, 1994; Giuntini and Greuling, 1989; Kôpka and Chovanec, 1994) and sequential effect algebras (Gudder and Greechie, in press-a,b) are algebraic systems that have recently been introduced to study the structure of unsharp quantum events. An *effect algebra* is a system $(E, 0, 1, \oplus)$ where $0, 1 \in E$ and $\oplus$ is a partial binary operation on E that satisfies the following conditions.

(3.1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $b \oplus a = a \oplus b$.
(3.2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(3.3) For every $a \in E$ there exists a unique $a' \in E$ such that $a \oplus a' = 1$.
(3.4) If $a \oplus 1$ is defined, then $a = 0$.

We define a partial order relation $\preceq$ on E by $a \preceq b$ if there exists $ac \in E$ such that $a \oplus c = b$. We write $a \perp b$ if $a \preceq b'$. It can be shown that $(E, \preceq, ')$ is a bounded orthoposet with least and greatest elements 0 and 1, respectively. Moreover, $a \oplus b$ is defined if and only if $a \perp b$.

Although there is a general theory of sequential effect algebras, we shall only be concerned with the commutative case here. A *commutative sequential effect algebra* (SEA) is a system $(E, 0, 1, \oplus, \circ)$ where $(E, 0, 1, \oplus)$ is an effect algebra and $\circ : E \times E \to E$ is a binary operation that satisfies the following conditions.

(3.5) For every $a, b \in E$, $a \circ b = b \circ a$.
(3.6) For every $a \in E$, $1 \circ a = a$.
(3.7) If $b \perp c$, then $a \circ b \perp a \circ c$ and $a \circ (b \oplus c) = a \circ b \oplus a \circ c$.
(3.8) For every $a, b, c \in E$, $a \circ (b \circ c) = (a \circ b) \circ c$.

There are many examples of SEAs (Gudder and Greechie, in press-a,b). However, for our present discussion, we are only interested in the example $[0, 1] \subseteq \mathbb{R}$. The unit interval $([0, 1], 0, 1, \oplus, \circ)$ is a commutative SEA where $a \oplus b$ is defined if $a + b \leq 1$ in which case $a \oplus b = a + b$ and $a \circ b = ab$ for all $a, b \in [0, 1]$. If E and F are SEAs, a map $\phi : E \to F$ is an *isomorphism* if ϕ is surjective and satisfies:

(3.9) $\phi(1) = 1$.
(3.10) $a \perp b$ if and only if $\phi(a) \perp \phi(b)$ and in this case $\phi(a \oplus b) = \phi(a) \oplus (b)$.
(3.11) $\phi(a \circ b) = \phi(a) \circ \phi(b)$ for every $a, b \in E$.

If there is a isomorphism from E to F we say that E and F are *isomorphic*. Isomorphic SEAs are indistinguishable as far as their SEA structure is concerned.

For $0 \leq \lambda \leq 1$, we define

$$\rho\lambda = \frac{(1-\lambda)}{2^{n-1}} P_0 + \frac{\lambda}{2^{n-1}} P_1 \in \mathcal{D}(\mathbb{C}^{2^n})$$

Then $p(\rho\lambda) = \mathrm{tr}(P_1 \rho\lambda) = \lambda$. For, $\rho, \sigma \in \mathcal{D}$ if $p(\rho) + p(\sigma) \leq 1$ we define

$$[\rho] \oplus [\sigma] = [\rho\, p_{(\rho)+p(\sigma)}]$$

We then have

$$p([\rho] \oplus [\sigma]) = p([\rho]) + p([\sigma])$$

Moreover, we define $[\rho] \circ [\sigma] = \mathrm{AND}\ ([\rho], [\sigma])$ and we have

$$p([\rho] \circ [\sigma]) = p([\rho]) p([\sigma])$$

We have thus defined the partial binary operation $\oplus$ and the binary operation $\circ$ on the QCL L.

Theorem 3.5. *The QCL* $(L, [\rho_0], [\rho_1], \oplus, \circ)$ *is a commutative SEA and the logic order* $\leq$ *coincides with the effect algebra order* $\preceq$*. Moreover,* $p : L \to [0, 1]$ *is an isomorphism.*

Proof: It is straightforward to show that the commutative and associative laws (3.1), (3.2) hold in L. To verify (3.3), note that $[\rho] \oplus [\rho]' = [\rho_1]$ and $[\rho]'$ is a unique element of L with this property. To verify (3.4), suppose that $[\rho] \oplus [\rho_1]$ is defined. Then $p(\rho) + 1 \leq 1$ that implies that $p(\rho) = 0$. Hence, $[\rho] = [\rho_0]$. We conclude that L is an effect algebra. If $[\rho] \leq [\sigma]$ then $p(\rho) \leq p(\sigma)$. Hence,

$$[\rho] \oplus [\rho_{p(\sigma)-p(\rho)}] = [\sigma]$$

so that $[\rho] \preceq [\sigma]$. Conversely, if there exists a $\delta \in \mathcal{D}$ such that $[\rho] \oplus [\delta] = [\sigma]$, then $p(\rho) + p(\delta) = p(\sigma)$ so that $p(\rho) \leq p(\sigma)$. Hence, $[\rho] \leq [\sigma]$. It follows that $\leq$ and $\preceq$ coincide. It is clear that (3.5) and (3.6) hold. To verify (3.7) suppose that $[\sigma] \perp [\delta]$. Then $p(\sigma) + p(\delta) \leq 1$ and we have

$$[\rho] \circ ([\sigma] \oplus [\delta]) = [\rho] \circ [\rho_{p(\sigma)+p(\delta)}] = [\mathrm{AND}\,(\rho, \rho_{p(\sigma)+p(\delta)})]$$

Now

$$p[\mathrm{AND}\,(\rho, \sigma)] + p[\mathrm{AND}\,(\rho, \delta)] = p(\rho)p(\sigma) + p(\rho)p(\delta) \leq 1$$

so that $[\mathrm{AND}\,(\rho, \sigma)] \oplus [\mathrm{AND}\,(\rho, \delta)]$ is defined. Moreover,

$$p(\rho)[p(\sigma) + p(\delta)] = p[\mathrm{AND}\,(\rho, \rho_{p(\sigma)+p(\delta)})]$$

Hence,

$$\begin{aligned}[\rho] \circ ([\sigma] \oplus [\delta]) &= [\mathrm{AND}\,(\rho, \rho_{p(\sigma)+p(\delta)})] = [\mathrm{AND}\,(\rho, \delta)] \oplus [\mathrm{AND}\,(\rho, \delta)] \\ &= [\rho] \circ [\sigma] \oplus [\rho] \circ [\delta]\end{aligned}$$

To verify (3.8), since

$$p(\mathrm{AND}\,(\rho, \mathrm{AND}\,(\sigma, \delta))) = p(\rho)p(\sigma)p(\delta) = p(\mathrm{AND}\,(\mathrm{AND}\,(\rho, \sigma), \delta))$$

we have that

$$[\rho] \circ ([\sigma] \circ [\delta]) = ([\rho] \circ [\sigma]) \circ [\delta]$$

It follows that L is a commutative SEA. To show that $p : L \to [0, 1]$ is an isomorphism, it is clear that $p([\rho_1]) = 1$ so (3.9) holds. Now $[\rho] \perp [\sigma]$ if and only if $p(\rho) + p(\sigma) \leq 1$ in which case

$$p([\rho] \oplus [\sigma]) = p([\rho]) \oplus p([\sigma])$$

so (3.10) holds. Also, $p([\rho] \circ [\sigma]) = p([\rho])p([\sigma])$ so (3.11) holds. Finally, given a $\lambda \in [0, 1]$ we have $p(\rho\lambda) = \lambda$ so that p is surjective. □

An element a of an effect algebra is *sharp* if $a \wedge a' = 0$. Since the only sharp elements of [0, 1] are 0 and 1, it follows from Theorem 3.1 that the only sharp elements of L are $[\rho_0]$ and $[\rho_1]$. We conclude that the QCL is a purely "fuzzy logic" all of whose elements are unsharp except for the trivial elements $[\rho_0]$ and $[\rho_1]$.

REFERENCES

Cattaneo, G., Dalla Chiara, M., Giuntini, R., and Leporini, R. (in press). An unsharp logic from quantum computation. *International Journal of Theoretical Physics*.

Dvurečenskij, A. and Pulmannová, S. (2000). *New Trends in Quantum Structures*, Kluwer, Dordrecht.

Foulis, D. J. and Bennett, M. K. (1994). Effect algebras and unsharp quantum logics. *Foundations of Physics* **24**, 1325–1346.

Giuntini, R. and Greuling, H. (1989). Toward a formal language for unsharp properties. *Foundations of Physics* **19**, 931–945.

Gudder, S. and Greechie, R. (2002). Sequential products on effect algebras. *Rep. Math. Phys.* **49**, 87–111.

Gudder, S. and Greechie, R. (in press-b). *Uniqueness and Order in Sequential Effect Algebras*.

Kôpka, F. and Chovanec, F. (1994). D-Posets. *Math. Slovaca* **44**, 21–34.

Nielsen, M. and Chuang, I. (2000). *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, England.

Pittenger, A. (2001). *An Introduction to Quantum Computing Algorithms*, Birkhäuser, Boston.